Group Theory

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Contents

Chapter 1

MAST20022

Groups

Definition: Group *A set G and a function $\star : G \times G \to G$ is called a group iff*

- $\forall g, h, k \in G;\quad (g \star h) \star k = g \star (h \star k)$ *(The group operation is associative)*
- $\exists e \in G,\ \forall g \in G;\quad g \star e = e \star g = g$ *(unital)*
- $\forall g \in G,\ \exists h \in G;\quad g \star h = h \star g = e$

Notes:

- The group is strictly the pair $(G, \star)$ however is often denoted by G alone.
- The group operation is often denoted by simply concatenating group elements $g \star h = gh$
- There is a whole hierarchy of algebraic objects that have different combinations of unit, associative and commutative (monoid, groupoid etc)

Definition: *A group is abelian iff the group operation is commutative i.e.*

$$\forall g, h \in G;\quad gh = hg$$

Lemma. *The groups identity is unique*

Proof. Let e, e' be identity elements

$$e = e'e = e'$$

Lemma. *Inverse elements are unique*

Proof. Suppose that given a $g \in G$ there are two $h, h' \in G$ such that $hg = h'g = e$ then

$$h' = (hg)h' = h(gh') = he = h$$

we note that in general $(ab)^{-1} = a^{-1}b^{-1}$

Lemma. *Multiplication by a group element is a group automorphism. i.e.*

$$\varphi_g : G \to G$$

$$x \mapsto gx$$

Is an isomorphism

Proof. Clearly $\varphi_g \circ \varphi_{g^{-1}} = \varphi_{g^{-1}} \circ \varphi_g = id_G$ so we have a bijection. This is a group homomorphism by the associativity axiom.

Note that there is nothing here essential about the multiplication from the left, and so the same argument proves right multiplication is similarly a group automorphism.

Element Order

We define the order of a group element by considering the set $S_g = \{n \in \mathbb{N} : g^n = e\}$, where g^n is notation for $g \star g \star ... \star g$ n times. This gives a family of homomorphisms

$$f_g : \mathbb{Z} \to G$$
$$n \mapsto g^n$$

We can denote the image of f_g to be $\langle g \rangle$, an abelian subgroup of G.

Definition:
$S_g = \emptyset \implies$ g has infinite order
$S_g \neq \emptyset \implies$ g has finite order

Definition: *If $g \in G$ has finite order then we denote*

$$|g| = min(S_g)$$

Alternatively $|g| = |\langle g \rangle|$

Lemma.

$$g^n = e \implies |g| \Big| n$$

Proof. If $g^n = e$ we get that $n \in Ker(f_g) = n_g\mathbb{Z}$ for some $n_g \in \mathbb{N}$ (all subgroups of $\mathbb{Z}$are cyclic). $n > 0$ implies that (we can choose) $n_g > 0$. But this $n_g = |g|$ and we have shown that $n \in n_g\mathbb{Z}$ i.e. $|g| = n_g|n$

Lemma.

$$h \in \langle g \rangle \implies |h| \Big| |g|$$

Proof. Using 1.2.2 and 1.1.1. Let $h \in \langle g \rangle$ then we know there is some $a \in \mathbb{N}$ such that $g^a = h$ thus

$$h^{|g|} = g^{|g|a} = e \implies |h| \Big| |g|$$

Subgroups

Definition: *A subset, $H \subseteq G$, is a subgroup if the operation on G restricted to H also forms a group. We denote this as*

$$H \leq G$$

Subgroup Spanned

Lemma.

$$\begin{aligned} H \leq G &\iff \forall x, y \in H; \quad xy,\ x^{-1} \in H \\ &\iff \forall x, y \in H; \quad xy^{-1} \in H \end{aligned} \tag{1.1}$$

Definition: *The subgroup generated by $S \subseteq G$ is*

$$\langle S \rangle = \bigcap_{H \leq G, S \subseteq H} H$$

One notes that $\langle \emptyset \rangle = \{e\}$

Cyclic Groups

Definition: *A group G is cyclic iff $\exists g \in G \quad G = \langle g \rangle$*

Lemma.

$$\langle g \rangle = \{g^n : n \in \mathbb{Z}\}$$

Lemma. *Every subgroup of a cyclic group is cyclic*

Proof. Let G be a cyclic group. If G is infinite then $G \cong \mathbb{Z}$ and thus every subgroup cyclic. Suppose G is finite with $|G| = N$. Then $G \cong \mathbb{Z}/N\mathbb{Z}$.

Lemma.

$$|g| = |\langle g \rangle|$$

i.e. the order of g is the cardinality of the subgroup it generates

Group Homomorphisms

Definition: *A group homomorphism is a function $\phi : G \to H$ between groups such that*

$$\phi(gh) = \phi(g)\phi(h)$$

The following are basic properties of homomorphisms

- $\phi(e_G) = e_H$
- $\phi(g^{-1}) = (\phi(g))^{-1}$
- $|g| \in \mathbb{N} \implies |\phi(g)| \in \mathbb{N}$ and $|\phi(g)| \Big| |g|$
- The inverse to a bijective homomorphism is also a homomorphism

Definition: *A bijective homomorphism is an isomorphism (of groups)*

An interesting example is the isomorphism given between $(\mathbb{R}, +)$ and $(R, \times)$ by $e^{(-)}$.

Theorem. *G a cyclic group*

- *G infinite* $\implies G \cong \mathbb{Z}$
- $|G| = m \in \mathbb{N} \implies G \cong \mathbb{Z}/m\mathbb{Z}$

Kernel and Image

Definition: *Let $\phi : G \to H$ be a homomorphism*

$$ker(\phi) = \{g \in G : \phi(g) = e_H\}$$

Definition:

$$Im(\phi) = \{\phi(g) : g \in G\}$$

Lemma.

$$im(\phi) \leq G$$

Lemma.

$$ker(\phi) \triangleleft G$$

Lemma.

$$\phi \text{ injective} \iff ker(\phi) = \{e\}$$

Proof. If ϕ is injective then $ker(\phi) = \{e\}$ by definition of homomorphism.
If $ker(\phi) = \{e\}$ and $\phi(x) = \phi(y)$ then

$$e = \phi(y)^{-1}\phi(x) = \phi(y^{-1})\phi(x) = \phi(y^{-1}x)$$

Thus $y^{-1}x \in ker(\phi)$ and so $y^{-1}x = e \implies x = y$.

Making New Groups

Direct Product

Given two groups $(G, \star), (H, \cdot)$ we construct a new group called their product with

- Set $G \times H$
- Component wise operation, $(\star, \cdot)$

$$(g_1, h_1)(g_2, h_2) = (g_1 \star g_2, h_1 \cdot h_2)$$

Quotient Group

Cosets

Consider a group G and a subgroup $H \leq G$. A left coset of H in G is given by (and a right coset is just on the other side)

$$gH = \{gh : h \in H\}$$

Note that H is itself a left and right coset in G because $H = eH = He$

Lemma.

$$g \notin H \implies gH \nleq G$$

Proof.

Like wise for right cosets.
We have the following basic properties of cosets: $\forall a, b \in G$

- $aH = bH \iff a^{-1}b \in H$ (same on the right)
- Left / right cosets partition G
- $f : aH \to bH, ah \mapsto bh$ is a bijection (same on the right)

Proof. Assume $aH = bH$. Then for some $h \in H$ $ah = be = b$ so $a^{-1}y = h \in H$. For the reverse assume $a^{-1}b = h \in H$ then $b = ah$ for some $h \in H$ and $aH \subseteq bH$ reversing the role of a and b we see that $aH = bH$.

(Of partition) It is clear that $g \in gH$ for every $g \in G$ (H is a subgroup so contains the identity) so $\Cup_{g \in G} gH = G$. Let $z \in g_1 H \cap g_2 H$ and assume $g_1 H \neq g_2 H$ then for some group elements $h_1, h_2 \in H$ we have that $z = g_1 h_1 = g_2 h_2$ but this implies that $g_2^{-1} g_1 = h_2 h_1^{-1} \in H$ thus by the first point $g_1 H = g_2 H$ a contradiction. So no such z can exist and we have shown $g_1 H \cap g_2 H = \emptyset$. Thus the cosets partition G.

Lemma.

$$\forall g \in G \ \ |gH| = |Hg| = |H|$$

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Definition: *The index of $H \leq G$ is*

$$[G : H] = |\{gH : g \in G\}| = |\{Hg : g \in G\}|$$

Theorem. *Lagranges Theorem:*

$$|G| \in \mathbb{N} \text{ and } H \leq G \implies |G| = |H|[G : H]$$

Proof. Let gH be a coset. We have that multiplication by a group element is a bijection hence $H \to gH$ is a bijection. We have that the left cosets partition G implying, thus $|G| = \sum_{gH \in G/H} |gH| = |H|[G : H]$

Where G/H denotes simply the set of cosets, this forms a group under the special case where H is a normal subgroup.

Lemma. *For a finite group G and $g \in G$*

- $g^{|G|} = e$
- $|g| \Big| |G|$

Lemma.

$$|G| = p \in \{q \in \mathbb{N} : \ q \text{ is prime } \} \implies G \cong \mathbb{Z}/p\mathbb{Z}$$

Normal Subgroup

Definition: *$H \leq G$ is a normal subgroup of G iff $\forall g \in G$ we have $gH = Hg$. We denote this $H \triangleleft G$*

Note that if G is abelian all subgroups are normal.

Quotient Group

Let G be a group and $H \triangleleft G$ then we form the quotient group of G by H, denoted G/H (left quotient) by

- The set is $G/H = \{gH : g \in G\}$
- Operation $(gH)(kH) = (g \star k)H$

Lemma. *When G is finite we get that*

$$|G/H| = |G|/|H|$$

First Isomorphism Theorem

There is a natural projection $\pi : G \to G/H, g \mapsto gH$. This map is a surjective homomorphism with $ker(\pi) = H$.

Theorem. *Let $\phi : G \to H$ be a homomorphisms*

$$G/ker(\phi) \cong Im(\phi)$$

Proof. Denote $K = ker(\phi)$ and $I = Im(\phi)$ The isomorphism is

$$\varphi : G/K \to I$$

$$gK \to \phi(g)$$

We must check that this is a well definined bijective group homomorphism.

Homomorphism is easily checked and surjectivity follows from the surjectivity of ϕ onto its image. Remains to check that it is a well defined injection.

$$\begin{aligned}\phi(x) = \phi(y) &\iff e = phi(y)^{-1}\phi(x) = \phi(y^{-1}x)\\ &\iff y^{-1}x \in K\\ &\iff xK = yK\end{aligned}$$

Which shows both injectivity and well definedness (iff's)

Group Actions

Let G be a gro up and X some set.

Definition: *A left action of G on X is a function* $\cdot : G \times X \to X$ *such that* $\forall x \in X, \forall gh \in G$:

- $e \cdot x = x$
- $(gh) \cdot x = g \cdot (h \cdot x)$

We denote the fact that G acts on X with $G \curvearrowright X$.

Orbit-Stabiliser

Definition: *Let* $G \curvearrowright X$ *and* $x \in X$

- *The orbit of x is* $O(x) = \{g \cdot x : g \in G\} = G \cdot x$
- *The stabiliser of x is* $stab(x) = \{g \in G : g \cdot x = x\}$
- *The stabiliser of* $S \subseteq X$ *is* $G_S = \{g \in G : gS = S\}$

Definition: x *is a fixed point of* $G \curvearrowright X$ *iff* $stab(x) = G$

Definition: *An action is transitive iff there is only one orbit*

Lemma. *Let* $G \curvearrowright X$ *then*

- $S \subseteq X$ *then* G_S *is a subgroup of G*

Lemma. *The orbits of* $G \curvearrowright X$ *partition X.*

Proof. Clearly $s \in Gs$ (look at the action of the identity) then we get that $X = \uplus_{x \in X} Gx$. We need only show that they are mutually disjoint.

Let $z \in Gs \cap Gt$ then there are $g_1, g_2 \in G$ such that $z = g_1 s = g_2 t$. Rearranging gives that $s = g_1^{-1} g_2 t$ and so $s \in Gs$. Thus $Gt \subseteq Gs$, but by symetry we also get that $Gs \subseteq Gt$ thus $Gs = Gt$.

Theorem.

$$G/stab(x) \xrightarrow{\sim} O(x), \quad gstab(x) \mapsto g \cdot x)$$

Proof. Routine using $sx = tx \iff sG_x = tG_x$

Theorem. *If G is finite and G $\curvearrowright$ X then*

$$|G| = |O(x)||stab(x)|$$

Proof.

$$|G/stab(x)| = |G|/|stab(x)| = |O(x)|$$

Because G is finite and we have the above bijection.

Definition: *If both X and G are finite sets then we denote the set of fixed points of $g \in G \curvearrowright X$ by*

$$X^g = \{x \in X : g \cdot x = x\}$$

Lemma. *Burnsides Orbit Counting Lemma:*

$$|\{O(x) : x \in X\}| = \frac{1}{|G|} \sum_{g \in G} |X^g|$$

Proof. Count in two ways

$$\sum_{x \in S} |G_x| = \sum_{orbits Gx} \sum_{y \in G_x} |G_y| = \sum_{orbits Gx} |G_x||Gx| = \sum_{orbits Gx} |G| = |S/G||G|$$

Center

Definition: *The center of a group G is*

$$Z(G) = \{g \in G : \forall h \in G, gh = hg\}$$

The center of a group is the equivilent to the set of fixed points to the action $G \curvearrowright G$ by conjugation.

Structural Theorem for Finitely Generate Abelian Groups

Examples

Symetric Groups

For a finite n we say that a permutation of $[n] := \{1, ..., n\}$ is a bijection

$$p : [n] \to [n]$$

We denote the collection of all permutations of $[n]$ by S_n. S_n forms a group with operation composition known as the symmetric group.

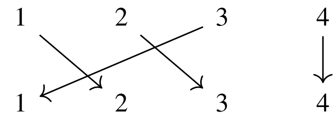
Cycle Notation

We can represent $\sigma \in S_n$ in the following compact notation:

- σ is represented as a sequence of tupples
- Each element in $[n]$ appears exactly once
- i is always followed by $\sigma(i)$ within a tuple and the last element maps back to the first (hence cycle)

- Note that singleton cycles may be ommited

  For example



can be represented as

$$(1\ \ 2\ \ 3)(4) = (1\ \ 2\ \ 3)$$

Note that disjoint cycles commute.

Isometry Group

An isometry is a distance preserving map (for a given metric space). Consider the set

$$L = \{\varphi : \mathbb{R}^2 \to \mathbb{R}^2 | \varphi \text{ is an isometry that preserves } (0,0)\}$$

It can be shown that every such map is infact both linear and invertible. This forms a group under composition.

Dihedral Group

For an $n \geq 3$ we can define the dihedral group D_n to be the group (under composition) of symmetries of a regular n-gon. If r represents a rotation by $\frac{2\pi}{n}$ and s a reflection through some fixed axis then

$$D_n = \{e, r, r^2, ..., r^{n-1}, s, rs, r^2 s, ..., r^{n-1} s\}$$

Alternatively the dihedral group is the free group on two generators with the following relations:

$$D_n = \langle r, s : r^n = e, s^2 = e, sr = r^{n-1} s \rangle$$

An isomorphism

It can be shown that $S_3 \cong D_3$.

Prime Groups

Throughout let p be a prime number.

Lemma.

$$|G| = p^n \implies |Z(G)| \geq p$$

Lemma. *Let G act on a finite set X.*

$$p \not| |X| \implies G \curvearrowright X \text{ has a fixed point}$$

i.e. there is some $x \in X$ such that the stabiliser of x is lal of G

Theorem. *Cauchys Theorem: For a finite group G*

$$p \Big| |G| \implies \exists g \in G, |g| = p$$

Sylow Theorems

Theorem. *Let G be a finite group*

$$p^s \Big| |G| \implies \exists H \leq G; |H| = p^s$$

i.e. if a prime power divides the cardinality of the group then there is a subgroup of cardinality that prime power.

A group of size (cardinality) p^s is called a p group.

Definition: *A Sylow-p subgroup, H, of a finite group, G, is a subgroup such that*

- $|H| = p^s$
- p^s *is the highest power of p dividing the order of G*

Theorem. *Any two Sylow p-subgroups are conjugate. i.e. For two Sylow p-subgroups $H, K \leq G$ there exists a $g \in G$ such that $gHg^{-1} = K$*

Theorem. *For a finite group G such that $p \Big| |G|$ then*

- $|\{ \text{Sylow p subgroups}\}| \Big| |G|$
- $|\{ \text{Sylow p subgroups}\}| \equiv 1$ *modulo p*

Theorem. *For a finite group G and a subgroup $H \leq G$. If H is a p group then there is a subgroup of H that is a Sylow p-subgroup.*

Matrix Groups

$GL_n(\mathbb{R})$

We call

$$GL_n(\mathbb{R}) = \{n \times n \text{ invertible matrices with entries in } \mathbb{R}\}$$

under matrix multiplication the general linear group. Note that this forms a group through the following observations

- The identity matrix is the group identity
- Every invertable matrix has an inverse
- $det(AB) = det(A)det(B)$
- $det(A^{-1}) = det(A)^{-1}$

Note that this is a non-abelian group of infinite cardinality.

Orthogonal Group

A subgroup of $GL_n(\mathbb{R})$ is the orthogonal group

$$O_n(\mathbb{R}) = \{M \in GL_n(\mathbb{R}) : MM^T = I\}$$

i.e. each matrix in the orthogonal group has its transpose as its own inverse.

In fact we have that $L \cong O_2$.

Chapter 2

AMSI Geometric Group Theory

Review

Group Examples

$\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$, vector spaces matrix groups, symmetric groups, dihedral groups, quaternions, $A_{\mathbb{N}}, S_{\mathbb{N}}$, rigid motions of the plane, symmetris of platonic solids, frieze groups, wallpaper groups, automorphisms of graphs.

Actions

There is an alternate definition of action

> ***Definition:*** *An action of G on X is a homomorphism $G \to Aut(X)$, where $Aut(X)$ is the group of bijections from X to itself (under compostion)*

We call an action $G \curvearrowright X$

- faithful iff the action has trivial kernel
- transitive iff $\forall x, y \in X$ there is a $g \in G$ $gx = y$
  iff there is only one orbit of G on X
- free iff the stabiliser of every $x \in X$ is trivial
- regular iff it is free and transitive

Moniods

For a set X we denote the cartesian product of X with itself n times by X^n. A word on the alphabet X is then an element of

$$X^* = \bigcup_{n \in \mathbb{N}} X^n$$

(note that $\mathbb{N}$ includes 0). The unique element of X^0 is the empty word denoted by ϵ (by convention we simply define $X^0 = \{\epsilon\}$).

> ***Definition:*** *A monid is a set with a binary operation satisfying closure, associativity and identity*

Cayley Graphs

Definition:

- *A **directed graph** $\Gamma = (V, E)$ consists of a set of verticies V, and a set of edges $E \subseteq V \times V$. An element of E will be (u, v) wich we interpret as an edge FROM u TO v.*
- *If $(u, v) \in E$ or $(v, u) \in E$ we say that u and v are **adjacent**.*
- *A **directed path** is a sequence of vertices, $(v_0, ..., v_n)$ such that $(v_i, v_{i+1}) \in E$ for each i.*
- *A directed cycle is a path such that $v_0 = v_n$*
- *A directed loop is a directed cycle of length one.*
- *A directed graph is **diconnected or strongly connected** if there is a finite directed path between every pair of vertices.*
- *We call the cardinality of the following set the indegree (respectively out degree) of a vertex u, $\{v \in V : (v, u) \in E\}$ (respectively $\{v \in V : (u, v) \in E\}$)*
- *A directed graph is locally finite if both in and out degree of every vertex is finite.*
- *An **undirected graph** (V, E) consists of a set of vertices and a set of edges $E \subseteq \{x \in \mathcal{P}(V) : 1 \leq |e| \leq 2\}$*
- *Vertices, u, v, of an undirected graph are adjacent iff $\{u, v\} \in E$*
- *The valency or degree of a vertex is the number of edges in which the vertex appears, plus one if the vertex has a loop (so loops contribute two to the degree).*
- *An undirected graph is locally finite iff every vertex has a finite degree.*
- *A **path** of length $n \in \mathbb{N} \cup \{\infty\}$ (in an undirected graph) is a sequence of vertices such that $\{v_i, v_{i+1}\} \in E$*
- *A path in a directed graph is a path in the underlying undirected graph.*
- *An undirected graph is **connected** if there is a finite path connecting any pair of vertices.*
- *A directed graph is connected if its underlying undirected graph is.*
- *A cycle in an undirected graph is a path such that $v_0 = v_n$.*
- *A loop is a cycle of length one.*
- *A cycle, $(v_0, v_1, ..., v_{n-1}, v_0)$, is non-trivial iff $n > 2$ and all the vertices $v_0, ..., v_{n-1}$ are distinct.*
- *A tree is an undirected graph which is connected and has no non-trivial cycles*
- *A **directed graph morphism** $\Gamma_1 = (V_1, E_1) \to \Gamma_2 = (V_2, E_2)$ is a map $\phi : V_1 \to V_2$ preserving adjacency i.e. $(u, v) \in E_1 \iff (\phi(u), \phi(v)) \in E_2$*
- *If the graphs Γ_1 and Γ_2 are also edgelabelled; i.e. have functions $\ell_1 : E_1 \to S$ (S a set of labels) an $\ell_2 : E_2 \to S$ then a label preserving directed graph morphism is a directed graph morphism preserving labelling, i.e. $\phi : V_1 \to V_2$ is such that $\ell_1(u, v) = \ell_2(\phi(u), \phi(v))$*

Definition: *For a group G and a subset $S \subseteq G$ we define the (right) Cayley graph of G with respect to (wrt) S to be*

$$Cay(G, S) = (V = G, E = \{(g, gs) : g \in G, s \in S\})$$

This also comes with a natural labeling map

$$\ell : E \to S$$
$$(g, h) \mapsto g^{-1}h$$

Notice that every vertex has valency $|S \cup S^{-1}|$.

Theorem. *Every group can be realised as a subgroup of a permutation (automorphism) group.*

Proof. For each element $g \in G$ we get a group automorphism $\lambda_g : G \to G, \quad h \mapsto gh$. Such an automorphism we call a permutation.

Thus there is a map $\lambda : G \to Aut(G)$. In fact λ is an injective homomorphism

$$\begin{aligned} \lambda(g) = \lambda(h) &\implies \forall k \in G, \quad gk = hk \\ &\implies ge = g = h = he \end{aligned}$$

Checking that it is a homomorphism is routine. Thus by the first isomorphism theorem

$$G/ker(\lambda) = G/e = G \cong \lambda(G) \leq Aut(G)$$

Because the image of a homomorphism is a subgroup.

This homomorphism λ is called the **left regular action** of G on itself.

Theorem. *Every group G generated by a finite set S acts faithfully on Cay(G, S), which is a locally finite, connected and directed graph.*

The action is regular on the verticies and free on the directed edges.

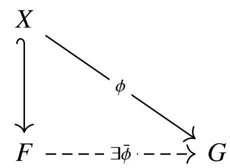
The action on undirected edges is free iff S contains on elements of order 2.

Theorem. *For any group G with a finite generating set S, the group fo label preserving directed graph automorphisms of $Cay(G, S)$ is isomorphic to G.*

Free Groups and Presentations

Definition:

- *A subset X of a group F is a free basis for F iff for every group G and every set function $\phi : X \to G$ there is a unique extension homomorphism $\bar{\phi} : F \to G$ such that $\bar{\phi}|_X = \phi$.*



- *A group is a free group iff it has a free basis*
- *The rank of a free group is the cardinality of its free basis*

Theorem. *Given any set X, there is a free group $F(X)$ with X as a basis.*

Proof. Let X be a set. Define $X^{-1} = \{x^{-1} : x \in X\}$, where x^{-1} is NOT the inverse of x (its just a set) but a notation for the corresponding element of X in X^{-1} (X^{-1} can just be any other bijective set).

Now consider the equivilence relation $\sim$ on $(X \sqcup X^{-1})^*$, the set of words over the disjoint union of X and its "inverse". $\sim$ is generated by $uxx^{-1}v \sim ux^{-1}xv \sim uv$ for any $x \in X, u, v \in (X \sqcup X^{-1})^*$.$F(X) = (X \sqcup X^{-1})^* / \sim$ is a group under concatenation.

Excercise: Check that $F(X) = (X \sqcup X^{-1})^* / \sim$ is a group under concatenation

We can identify X with its image under a suitable inclusion into F(X), $x \mapsto [x]$. X is a free basis of F(X).

So let a group G and a set map $\phi : X \to G$ be given. We construct a lift by letting $\bar{\phi} : F(X) \to G$ be the map sending $x_1^{\pm 1} \dots x_n^{\pm 1} \mapsto \phi(x_1)^{\pm 1} \dots \phi(x_n)^{\pm 1}$. One sees imediately that $\bar{\phi}|_X = \phi$, then uniqueness and homomorphism are a routine check.

Its in the notes, type up or do it as an exercise

$F(X)$ is the free group generate on X.

Lemma. *Every group is a quotient of a free group*

Proof. Let G be an arbitrary group. Then let $\phi : G \to G$ be the identity.

G is a free basis of $F(G)$ the free group generated on G (considered as a set) and so there is a unique extension of ϕ, $\bar{\phi} : F(G) \to G$. Moreover, this is clearly a surjection, thus by the first isomorphism theorem

$$G \cong F(G)/ker(\bar{\phi})$$

Note that any generating set of G, say X will actually suffice, if we let $\iota : X \to G$ be the inclusion

$$G \cong F(X)/ker(\bar{\iota})$$

Normal Form

The construction of the free group generated on a set naturally leads to a concept of normal form, a unique way of representing each element of the free group (a distinguished member of the equivilence class).

A word in $(X \sqcup X^{-1})^*$ is reduced iff it does not contain a subword of the form xx^{-1} or $x^{-1}x$. Any unreduced word is equivilent (under $\sim$ as defined in 2.3) to unique reduced word.

We define the length of the word $x_1 ... x_n \in (X \sqcup X^{-1})^*$ to be n.

In other words the reduction relation on words has the Church-Rosser property and is strongly normalising.

Lemma. *For any set X and any* $[u] \in (X \sqcup X^{-1})^*/\sim$ *there is a unique* $v \in [u]$ *that is reduced.*

Proof. Because the words are always of finite length it is obvious that we can reduce the words by simply iteratively removing occurences of xx^{-1} and $x^{-1}x$ until there are none. (Note that this strictly decreases the size of the word at each step so must terminate).

It remains to show that this normal form is unique. So suppose we have $v, w \in [u]$ two reduced words. By definition of the equivilence relation there is a finite sequence of words $w = u_0, ..., u_n = v$ such that u_i is obtained from u_{i-1} by the insertion or deletion of a term of the form xx^{-1} or $x^{-1}x$.

Let $L = \sum_{i=1}^{n-1} |u_i|$, the sum of the lengths of words in such a sequence. Now choose a sequence that minimises L. We want to show that L must be zero.

Assume that $L \neq 0$:

Both w and v are reduced so we have that $|w| < |u_1|$ and $|v| < |u_{n-1}|$ (equal would mean adding and removing a xx^{-1} or $x^{-1}x$ pair, while if it was less then there would be some reduction step that could be taken in w and v hence they would not be reduced). Thus the chain of lengths given by $|u_1|, ..., |u_{n-1}|$ cannot be strictly increasing or decreasing, so there is some i such that $|u_{i-1}| < |u_i| > |u_{i+1}|$. This says that $u_{i-1} \to u_i$ is an insertion and $u_i \to u_{i+1}$ is a deletion. This gives us two cases:

Case 1: Insert and delete different terms: If they are different terms we have something of the form

$$u_{i-1} = t_1 t_2 yy^{-1} t_3 \to u_i = t_1 xx^{-1} t_2 yy^{-1} t_3 \to u_{i+1} = t_1 xx^{-1} t_2 t_3$$

Where there are several cases (contrast $x^{-1}x$, or x being to the right of the yy^{-1} pair) that are essentially the same. This lets us create a new sequence by first deleting and then inserting, making the sequence

$$u_{i-1} = t_1 t_2 yy^{-1} t_3 \to u_i = t_1 t_2 t_3 \to u_{i+1} = t_1 xx^{-1} t_2 t_3$$

hence the value of L is now 4 lower. This is a contradiction (of minimality of L).

Case 2: Insert and delete the same term: This implies that $u_{i-1} = u_{i+1}$, hence we can remove the u_i term from the sequence again contradicting the minimality of L.

Thus $L = 0$ and $w = v$.

Thus the equivilence classes of $F(X)$ are often identified with their reduced words. Moreover any element of $F(X)$ can be uniquely written in the form

$$x_1^{r_1} \dots x_m^{r_m}$$

where $r_i \in \mathbb{Z} \setminus \{0\}$ and $x_i \neq x_{i+1}$

Lemma. *Let G be a group and* $X \subseteq G$*. G is free with basis X iff each element* $g \in G$ *can be written uniquely as*

$$x_1^{r_1} \dots x_m^{r_m}$$

where $r_i \in \mathbb{Z} \setminus \{0\}$ *and* $x_i \neq x_{i+1}$

Proof as exercise

We denote F_n the free group on n generators.

Lemma. F_2 *contains* F_3 *as a finite index subgroup.*

Proof as exercise; kind of step by step help given.

In fact F_2 has a subgroup isomorphic to F_k for any $k \in \mathbb{N}_{\geq 3}$.

Theorem. *Every subgroup of a free group is free.*

The proof is delayed.

Ping Pong

Theorem. *Let G be a group with generating set S which contains at least two elements s, t such that $s \neq t^{\pm}$. Now suppose G acts on a set X. If*

- *for each $r \in S$ there is nonempty subset $X_r \subseteq X$ such that the X_i's are pairwise disjoint*
- *$s^k \cdot X_r \subseteq X_s$ for every $s \neq r^{\pm 1}$ and every $k \in \mathbb{Z} \setminus \{0\}$*

Then G is free with basis S

Proof. (sketch.)
Assume the hypotheses of the Lemma. By 2.3.1 we just need to show that each element of G can be uniquely written as a product of powers of elements of S, or equivalently that a freely reduced word over $S \cup S^{-1}$ of length at least one is not the trivial element (this is injectivity in 2.3.1, where surjectivity is by the hypothesis of S being a generating set).
So take a $w = w_1^{k_1} \cdots w_n^{k_n}$ a freely reduced word where each $w_i \in S$ and $k_i \in \mathbb{Z} \setminus \{0\}$. There is some delicacy with the case were S has exactly two elements, however this is a sketch so we assume that we can choose $r \in S$ such that $r \neq w_1, w_2$. Then use the action to see that w is not the identity

$$w \cdot X_r \subset X_{w_1}$$

Which by hypothesis is disjoint from X_r so $w \neq 1$.

Presentations

Definition:

- *Let a set X be given and an $R \subseteq F(X)$. The normal closure of R in F(X), which we denote $\langle R \rangle_{F(X)}$ is the subset of F(X) comprising elements of the form*

$$\prod_{i=1}^{p} c_i^{-1} r_i^{\epsilon_i} c_i$$

where $p \in \mathbb{N}, c_i \in F(X), r_i \in R$ and $\epsilon = \pm 1$

- *A presentation is a set X and a subset $R \subseteq (X \cup X^{-1})^*$. We denote this as $\langle X|R \rangle$*
- *A presentation determines a group $F(X)/\langle R \rangle$*
- *A group is finitely presentable iff it admits a presentation ($\exists$) in which both X and R are finite.*

Lemma. *$\langle R \rangle$ is a normal subgroup of $F(X)$*

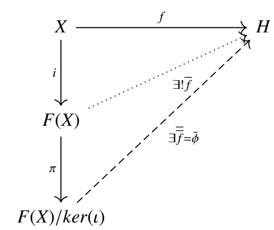
Proof is exercise

Lemma. *$H \triangleleft F(X)$ and $R \subseteq H \implies \langle R \rangle \triangleleft H$*

Proof is exercise

Lemma. *(Von Dycks Theorem) Let $G = \langle X|R \rangle$ and H be a group. A function $\phi : X \to H$ extends to a homomorphism $\bar{\phi} : G \to H$ iff whenever $x_1, ..., x_n \in X$ and $\epsilon_i = \pm$ such that $x_1^{\epsilon_1} \cdots x_n^{\epsilon_n} \in R$ we have that*

$$\phi(x_1)^{\epsilon_1} \cdots \phi(x_n)^{\epsilon_n} =_H 1$$



Note that this is effectively just stacking the universal property of the quotient ontop of the universal property of the free group. Where the condition given is exactly that the homomorphism extending from the free group factors through the quotient and hence extends using the universal property of the quotient.

Note that we check the equality for all expressions that form relators (the x_i are not unique necissarily), the lemma is just telling us how we need to check.

Tietze Transformations

Tietze transformations are transformations on a group presentation that do not chagne the group.

Definition: *Given a group presetnation $\langle X|R\rangle$ then the following are Tietze transformations*

1. *Replace $\langle X|R\rangle$ by $\langle X|R \cup S\rangle$ for some $S \subset \langle R\rangle$*
2. *Replace $\langle X|R\rangle$ by $\langle X|R \setminus S\rangle$ for some $S \subset \langle R \setminus S\rangle$*
3. *Replace $\langle X|R\rangle$ by $\langle X \cup Y|R \cup S\rangle$ where $Y \cap F(X) = \emptyset$ is a set of new symbols, corrosponding to each is a word $u_y \in (X \cup X^{-1})^*$ and $S = \{y^{-1}u_y : y \in Y\}$*
4. *Replace $\langle X|R\rangle$ by $\langle X \setminus Y|R \setminus S\rangle$ where $Y \subseteq X$ such that $S = \{y^{-1}u_y : y \in Y\} \subseteq R$ with $u_y \in ((X \setminus Y) \cup (X \setminus Y)^{-1})^*$*

Theorem. *$\langle X|R\rangle \cong \langle Y|S\rangle$ iff there is a sequence of Tietze transformations taking one to the other.*
If the presentations are finite then finitely many transformations are required.

Definition: *A group is linear iff it is a subgroup of a group of matrices over a field.*

Lemma. *F_n is linear.*

Proof. Argument is that F_2 is linear. Subgroups of linear groups are linear. Every finite rank free group is a subgroup of F_2.

Word Metric and Coarse Geometry

In this section we consider a group G be a group with a finite and symetric (closed under inverse) generating set S.

Definition:

- *The word length of an element $g \in G$ with respect to S is defined as*

$$||g||_S = min\{n \in \mathbb{N} : g = s_1 \cdots s_n, s_i \in S\}$$

  or 0 *if* $g = 1$

- *The word metric is defined as*

$$d_S(g,h) = ||g^{-1}h||_S$$

- *A group G acts by isometries on a metric space (X, d) iff there is a homomorphism $\rho : G \to isom(X)$.*

- *A function $f : (X, d_X) \to (Y, d_Y)$ between metric spaces is*
  - *an isometric embedding iff $\forall x, \in X\ d_Y(fx, fy) = d_X(x, y)$*
  - *an isometry iff it is a surjective isometric embedding*
  - *a $\lambda - biLipschitz$ embedding iff*

$$\frac{1}{\lambda}d_X(x_1, x_2) \le d_Y(fx_1, fx_2) \le \lambda d_X(x_1, x_2)$$

  - *a $\lambda - biLipschitz$ equivilence iff it is a surjective $\lambda - biLipschitz$ embedding*
  - *For a given $\lambda \ge 1, \epsilon \ge 0$ f is a $(\lambda, \epsilon) - quasi - isometric$ embedding iff*

$$\frac{1}{\lambda}d_X(x_1, x_2) - \epsilon \le d_Y(fx_1, fx_2) \le \lambda d_X(x_1, x_2) + \epsilon$$

  - *a quasi-isometry iff it is a quasi-isometric embedding and there is some $C > 0$ such that*

$$\forall y \in Y\ \exists x \in X\ d_Y(fx, y) \le C$$

Lemma. *Let S_1, S_2 be two finite symetric generating sets for a group G.*

$$(G, d_{S_1}) \underset{biLipshitz}{\sim} (G, d_{S_2})$$

Proof. Let $\lambda_1 = max\{||x||_{S_2} : x \in S_1\}$ and $\lambda_2 = max\{||x||_{S_1} : x \in S_2\}$ (bounded sets becasue S_1 is finite). Then we see that

$$d_{S_2}(g,h) = ||g^{-1}h||_{S_2} \le \lambda_1 ||g^{-1}h||_{S_1} = \lambda_1 d_{S_1}(g,h)$$
$$d_{S_1}(g,h) = ||g^{-1}h||_{S_1} \le \lambda_2 ||g^{-1}h||_{S_2} = \lambda_2 d_{S_2}(g,h)$$

check why the inequality holds

therefore with $\lambda = max\{\lambda_1, \lambda_2\}$ and $f = id_G : G \to G$ we have that

$$\frac{1}{\lambda}d_{S_1}(g,h) \le d_{S_2}(id(g), id(h)) \le \lambda d_{S_1}(g,h)$$

Thus we have produced a value of λ and a function f that is a biLipschitz equivalence, therefore the metric spaces are biLipschitz equivalent.

Lemma. *The following are all quasi-isometric*

$$(Cay(G, S), d_S) \sim (G, d_S) \sim (G, d_T) \sim (Cay(G, T), d_T)$$

Lemma. *Given two q.i. finitely generated groups. Then one is finitely presentable iff the other is.*

Milnor-Schwartz

Definition:

- *A geodesic segment for a metric space (X,d) is an isometric embedding $\gamma : [a,b] \hookrightarrow X$*
- *Metric space X is geodesic iff for any $x, y \in X$ there is some geodesic segment $\gamma : [0, d(x,y)] \to X$ such that $\gamma(0) = x$ and $\gamma(d(x,y)) = y$*
- *A metric space is proper iff closed balls are compact*
- *A group, G, acts properly discontinuously on a metric space, X iff for every compact set $K \subseteq X$ $\{g \in G : gK \cap K \neq \emptyset\}$ is a finite set.*
- *$G \curvearrowright X$ cocompactly iff there is some compact set $K \subseteq X$ such thhat*

$$X = \bigcup_{g \in G} gK$$

- *$G \curvearrowright X$ geometrically iff it acts both properly discontinuously and cocompactly by isometries.*

Theorem. *If a group G acts geometrically on a proper geodesic metric space then G is finitely generated and for any $x_0 \in X$ the orbit map*

$$G \to X$$
$$g \mapsto gx_0$$

is a quasi-isometry.

Proof. G acts cocompactly so there is some compact $K \subseteq X$ such that $X = \cup_{g \in G} gK$, but compact is always bounded (sequential compact implies bounded) so there is some $R > 0$ and some $x_0 \in X$ such that $K \subseteq B(x_0, R) = B$, and therefore $X = \cup_{g \in G} gB(x_0, R)$. It is important here that X is proper becuase this gives us that B is also compact.

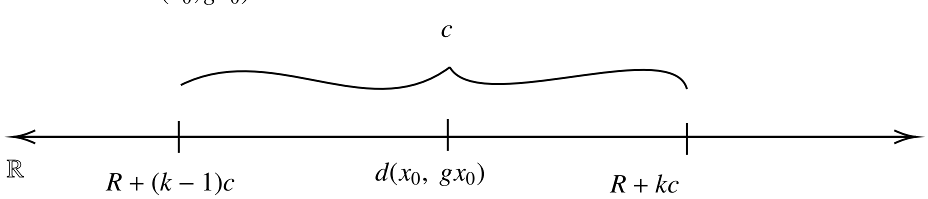
We define $S = \{g \in G : gB \cap B \neq \emptyset\}$ which is a finite and symmetric subset of G because B is compact and G acts properly discontinuously.

Claim: S generates G: Let $c = inf\{d(x, gy) : x, y \in B, g \in G\backslash\{1\}, g \notin S\} = inf\{d(B, gB) : g \in G\backslash\{1\}, g \notin S\}$. i.e. the smallest distance between any two points in any two disconnected balls. This $c > 0$. This is because take any gB disjoint from B, then there are finitely many translates of B that have a distance less than $d(B, gB)$ from the properly discontinuity of the action. Hence c is actually the minimum of finitely many positive numbers and hence positive.

Now consider some $g \in G \setminus S$ (in particular g is not 1). Then $d(x_0, gx_0) > R + c$ by definition of c and R. There is some constant $k \geq 2$ such that

$$R + (k-1)c < d(x_0, gx_0) \leq R + kc$$

basically because wherever $d(x_0, gx_0)$ sits on the real line there will be a number on either side that is c appart



Now let γ be a geodesic segment from x_0 to gx_0 (one exists because X is geodesic) and let $x_0, ..., x_k, x_{k+1} = gx_0$ be a sequence of points in X laying on the geodesic segment (in its image) such that $d(x_0, x_1) < R$, $d(x_i, x_{i+1}) < c$ for the other values of i. There is some sequence $1 = g_0, ..., g_{k-1}, g_k = g$ such that $x_{i+1} \in g_i B$ then so we define $s_i = g_{i-1}^{-1} g_i$ for $1 \leq i \leq k$.

$$\begin{aligned} d(B, s_i B) &= d(B, g_{i-1}^{-1} g_i B) \\ &= d(g_{i-1} B, g_i B) \\ &\leq d(x_i, x_{i+1}) \\ &< c \end{aligned}$$

So we can conclude that $s_i \in S$ because c was the minimum distance between the images of B when acted on by elements of $G \setminus S$.

Finally we see that $g = 1^{-1}g = (g_0^{-1}g_1)(g_1^{-1}g_2)\cdots(g_{k-1}^{-1}g_k) = s_1 \cdots s_k$. Thus S generates G.

Claim: The Orbit map is a quasi-isometry: Let $L = max\{d(x_0, sx_0) : s \in S\}$, $\lambda = max\{\frac{1}{c}, L, 2R\}$ and $\epsilon = max\{\frac{1}{\lambda}, c\}$. Now we show that g is a (λ, ϵ)-quasi-isometric embedding that is R-onto.

So for any $g \in G$ we have

$$\frac{1}{\lambda} d_S(1, g) - \epsilon \leq d(x_0, gx_0) \leq \lambda d_S(1, g)$$

i.e. the orbit map is a quasi-isometric embedding

show this as exercise

It remains to show the boundedness condition. For $g \in G \setminus S$ and $k \geq 2$ we can use the inequalities from the first claim to see that

$$R + (d_S(1, g) - 1)c < d(x_0, gx_0)$$

because $d_S(1, g) \leq k$, which can be deduced from the fact that $\frac{1}{\lambda} d_S(1, g) - \epsilon \leq R + (k-1)c$ which follows from the definition of λ and ϵ.

Rearrange to get that $cd_S(1, g) - c < d(x_0, gx_0) - R \leq d(x_0, gx_0)$. Now suppose $d_S(1, g) = n$ i.e. there is some $s_1, ..., s_n$ such that $g = s_1 \cdots s_n$ then

$$\begin{aligned}
d(x_0, gx_0) &= d(x_0, s_1 \cdots s_n x_0) \\
&\leq d(x_0, s_1 x_0) + d(s_1 x_0, s_1 \cdots s_n x_0) && \text{(triangle inequality)} \\
&\leq d(x_0, s_1 x_0) + ... + d(s_1 \cdots s_{n-1} x_0, s_1 \cdots s_n x_0) \\
&\leq \sum_{i=1}^{n} d(x_0, s_i x_0) && \text{(G acts by isometries)} \\
&\leq nL = Ld_S(1, g)
\end{aligned}$$

Lemma. *Let H be a finite index subgroup of G, a finitely generated group. Then H is finitely generated and G is quasi-isometric to H.*

Growth

Definition: *Throughout we will consider a finitely generated group G with finite generating set S*

- *The ball of radius n in the Cayley graph is*

$$B_{G,S}(n) = \{g \in G : d_S(1, g) \leq n\}$$

- *The growth function of G wrt to S is*

$$\begin{aligned} f_{G,S} &: \mathbb{N} \to \mathbb{N} \\ n &\mapsto |B_{G,S}(n)| \end{aligned}$$

- *Given $f, g : \mathbb{N} \to \mathbb{N}$ $f \preceq g$ iff $\exists C > 0$ such that $\forall n \in \mathbb{N}$ we have that $f(n) \leq Cg(Cn)$*
- $f \sim g \iff f \preceq g \wedge g \preceq f$
- *We denote γ_G to be any representative of the equivilence class of the growth function of G*
- *The exponential growth rate of a group G wrt S is $\omega(G, S) = \lim_{n\to\infty} (f_{G,S}(n))^{\frac{1}{n}}$*
- *G has exponential growth rate iff $\omega(G, S) > 1$*

- *G has subexponential growth iff* $\omega(G, S) = 1$
- *G has polunomial growth iff for some* $\alpha > 0$ $\gamma_G \leq n^\alpha$
- *G has intermediate growth iff it has subexponential and not polynomial growth.*
- *The commutator of two group elements* $g, h \in G$ *is*

$$[g, h] = ghg^{-1}h^{-1}$$

- *The commutator subgroup of G is the subgroup generated by all of its commutators. We denote it* $[G, G]$
- *Given* $H, K \leq G$ *subgroups we define* $[H, K]$ *to be the subgroup of G generated by*

$$\{[h, k] : h \in H, k \in K\}$$

- *The lower central series of G is* $G = G_0 \triangleright G_1 \triangleright \ldots$ *such that* $G_{i+1} = [G_i, G]$
- *G is nilpotent iff its lower central series terminates at* $\{1\}$ *in a finite number of steps*
- *Given a generating set S of G we say that a set of words in* $(S \cup S^{-1})^*$ *which is in bijection with G is a normal form.*
- *The derived series of G is*

$$G \triangleright [G, G] \triangleright [[G, G], [G, G]] \triangleright \ldots$$

- *G is solvable iff its derived series terminates at* $\{1\}$ *after finitely many steps.*
- *G is virtually nilpotent iff it contains a finite index nilpotent subgroup*

Lemma. *For any finitely generated group with generating set S*

$$f_{G,S}(n) \leq |S|(|S| - 1)^{n-1}$$

Proof. A finitely generated free group is the quotient of a finite rank free group. Thus every f.g. group grows slower than some finite rank free group. In this case the free group is on finite set S, hence in the first step there are $|S|$ choices and in all other steps there is $|S| - 1$ choices. Therefore the growth of $F(S)$ is

$$f_{F(S),S}(n) = |S|(|S| - 1)^{n-1}$$

and $f_{G,S} \leq f_{F(S),S}$ because it is a quotient.

This implies that finitely generated groups have at most exponential growth.

Lemma. *For any group, G, generated by the finite set S we have that*

$$\forall m, n \in \mathbb{N} \quad f_{G,S}(m + n) \leq f_{G,S}(m) f_{G,S}(n)$$

This is the property of being submultiplicative.

Proof. There are $f_{G,S}(m + n)$ words of length $m + n$, however any word of this length is precisely a word of length m with a word of length n concatenated on to it. There is at most $f_{G,S}(m)$ words of length m and at most $f_{G,S}(n)$ words of length n and so there can be at most $f_{G,S}(m) f_{G,S}(n)$ words of length $m+n$ (thinking combinatorially)

Lemma. *If G is infinite and finitely generated then* $f_{G,S}$ *is monotone increasing*

Proof. If a vertex can be reached in a path of length $\leq n$ it can certainly be reached in a path of length $\leq n + 1$. Finitely generated ensures the graph is locally finite and infinitude ensures that there are always more vertices to be reached (i.e. the growth function $f_{G,S}$ does not plateau for any n).

Lemma. *If G has two finite generating sets* S_1, S_2 *and we set* $M = max\{\|x_2\|_{S_1} : x_2 \in S_2\}$ *then*

$$f_{G,S_1}(n) \leq f_{G,S_2}(Mn)$$

Lemma. *If G is finitely generated by S and H finitely generated by T then*

$$G \underset{q-i}{\sim} H \implies f_{G,S} \sim f_{H,T}$$

i.e. $G \underset{q-i}{\sim} H \implies \gamma_G \sim \gamma_H$

proof exercise

Lemma. $\forall \alpha, \beta > 0$

- $n^\alpha \nsim n^\beta$
- $n^\alpha \nsim e^n$
- $\alpha^n \sim \beta^n$

Proof as exercise

So the equivilence on functions so defined differentiates well between polynomials, polynomials and exponentials but it cannot tell the difference between different exponentials.

Using the structural theorem for finitely generated abelian groups we have that $G \cong \mathbb{Z}^k \oplus$ (finite abelian groups) thus we have that $G \underset{q-i}{\sim} \mathbb{Z}^k$.

Lemma.

$$\gamma_G \sim \gamma_{\mathbb{Z}^k} \sim n^k$$

Lemma. *Finitely generated nilpotent groups have a groth function* n^k *for some* $k \in \mathbb{N}$.

Theorem. *A finitely generated group has polynomial growth iff it has a finite index nilpotent subgroup.*

Lemma. *The following are all sufficient conditions for G to have exponential growth rate*

- *G contains a free submonoid of rank 2*
- *There is a surjection* $G \to F_2$
- *There is a finite index subgroup H and a surjection* $H \to F_2$

Theorem. *For a finitely generated nilpotent group G, with lower central series* $G = G_0 \triangleright G_1 \triangleright \ldots \triangleright G_k = \{1\}$ *then* $\gamma_G \sim n^d$ *where d is given by*

$$d = \sum_{i=1}^{k-1} i \; rank(G_i/G_{i+1})$$

Where the rank is the rank of the torsion free part of these groups. Note that we apply here the structural theorem for finitely generated abelian groups. So each group is the direct sum of $\mathbb{Z}^n$ and some torsion groups. We can do this because we are quotienting groups by a commutator in the first step hence they are abelian.

Lemma. *If G is finitely generated with polynomial growth then G has a finite index nilpotent subgroup.*

Lemma. *A finitely generated and solvable group has either exponential growth or a finite index nilpotent subgroup*

Theorem. *For a finitely generated subgroup of* $GL_n(\mathbb{R})$ *we have either*

- *G has a finite index solvable group*
- *G contains* F_2

Dehn Functions

Definition:

- *If $\mathcal{P} = \langle S|R\rangle$ is a finite presentation of G and $w \in (S \cup S)^*$ which is equal to 1 in G then there are $N \in \mathbb{N}, r_i \in R, \epsilon_i = \pm 1, u_i \in F_S$ such that*

$$w =_{F_S} \prod_{i=1}^{N} u_i^{-1} r_i^{\epsilon_i} u_i$$

The area of w with respect to this presentation is then the minimum $N \in \mathbb{N}$ such that w has such a representation.

$$\mathcal{A}_{\mathcal{P}}(w)$$

- *The Dehn function of such a presentation is*

$$\delta_{\mathcal{P}} : \mathbb{N} \to \mathbb{N}$$

$$n \mapsto max\{\mathcal{A}_{\mathcal{P}}(w) : w \in (S \cup S^{-1})^*, w =_G 1, |w|_S \leq n\}$$

where $|w|_S$ is just the length of the string w.

- *Given $f, g : \mathbb{N} \to \mathbb{N}$ we say that $f \prec g$ iff there is some $C \geq 0$ such that $f(n) \leq Cg(Cn) + Cn$ for all $n \in \mathbb{N}$.*
- *$f \simeq g$ iff $f \prec g$ and $g \prec f$*
- *Two quasi-isometries, f, f', are $\mu - quasi - inverse$ iff $d_S(g, (f' \circ f)(g)), d_{S'}(g', (f \circ f')(g')) \leq \mu$ for all relevant g, g'*

Lemma. *Given two presentations $\mathcal{P} = \langle S|R\rangle, \mathcal{P}' = \langle S'|R'\rangle$ for G, G' respectively then*

$$G \underset{q.i.}{\simeq} G' \implies \delta_{\mathcal{P}} \simeq \delta_{\mathcal{P}'}$$

Proof. (sketch.)

It suffices to prove that $\delta_{\mathcal{P}} \prec \delta_{\mathcal{P}'}$ because we can just interchange their roles.

Now we use the quasi-isometrie between G, G' to get two $f : G \rightleftarrows G' : f'$ with constants λ, ϵ, μ such that f, f' are $(\lambda, \epsilon)-$quasi-isometries and are $\mu-$quasi-inverses.

Given an $n \in \mathbb{N}$ let $w = s_1 \cdots s_n \in (S \cup S^{-1})^*$ be the element realising the value of $\delta_{\mathcal{P}}(n)$, i.e. $w =_G 1,\ |w|_S = n$ and $A_{\mathcal{P}}(w) = \delta_{\mathcal{P}}(n)$. (If the word realising the Dehn function here has length less than n then the argument still goes through because we are bounding above.)

Denote w_i the elements of G which are determined by the first i characters of w. e.g. $w_0 = 1, w_2 = s_1 s_2, w_n = w$ etc. Then let $w_i' = f(w_i) \in G'$ and $v_i = f'(u_i') \in G$ (for some u').

We now choose geodesic segments between the consecutive w_i' to obtain a loop in $Cay(G', S')$ (loop becasue w_0 and w_n reduce to 1 by assumption) labelled by some $w' \in (S' \cup (S')^{-1})^*$. We can bound the length of this loop by

$$\begin{aligned} |w'| &= \sum^{n} d(w_i', w_{i-1}') && \text{(geodesic)} \\ &\leq nd(w_j', w_{j-1}') && \text{(take the maximum)} \\ &\leq n(\lambda d(w_i, w_{i-1}) + \epsilon) && \text{(q.i.)} \\ &= n(\lambda + \epsilon) && \text{(by construction)} \end{aligned}$$

Hence the area is bounded $A_{\mathcal{P}'}(w') \leq \delta_{\mathcal{P}'}(n(\lambda + \epsilon))$.

Now map back to G. Let $L' = max\{|r'|_{S'} : r' \in R'\}$ (the longest relator). Using the above calculation and noting that a relator is a loop in the cayley graph we can use the above bound (now in G) to see that for any relator mapped over to G in the same way (prefix peicewise and joinded with geodesics) we also get a loop in $Cay(G, S)$ with bounded length by $L'(\lambda + \epsilon)$ (L' is the max of any relator) and the area is bounded by $\delta_{\mathcal{P}}(L'(\lambda + \epsilon))$.

So by maping w' back to G we get a loop w'' in $Cay(G, S)$ of area bounded by

$$A_{\mathcal{P}}(w'') \leq \delta_{\mathcal{P}'}(n(\lambda + \epsilon))\delta_{\mathcal{P}}(L'(\lambda + \epsilon))$$

Finally "filling in" the area between the loops w, w'' we get

$$\begin{aligned}\delta_{\mathcal{P}}(n) = A_{\mathcal{P}}(w) &\leq A_{\mathcal{P}}(w'') + n\delta_{\mathcal{P}}(1 + 2\mu + \lambda(\lambda + \epsilon) + \epsilon)\\ &\leq \delta_{\mathcal{P}'}(n(\lambda + \epsilon))\delta_{\mathcal{P}}(L'(\lambda + \epsilon)) + n\delta_{\mathcal{P}}(1 + 2\mu + \lambda(\lambda + \epsilon) + \epsilon)\\ &\leq C\delta_{\mathcal{P}'}(Cn) + Cn\end{aligned}$$

where $C = max\{\lambda + \epsilon, \delta_{\mathcal{P}}(L'(\lambda + \epsilon)), \delta_{\mathcal{P}}(1 + 2\mu + \lambda(\lambda + \epsilon) + \epsilon)\}$

Integral Heisenberg Group

An important example that we will apply some of the established concepts to. Most of this takes place in the exercises. Maybe once they are done move them into this section.

$$H = \left\{\begin{pmatrix}1 & x & z\\ 0 & 1 & y\\ 0 & 0 & 1\end{pmatrix} : x, y, z \in \mathbb{Z}\right\}$$

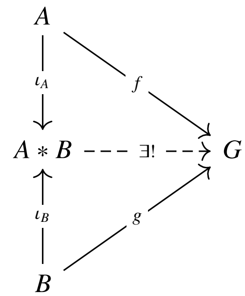
Which is a subgroup of $SL_3(\mathbb{Z})$, i.e. under matrix multiplication.

H is nilpotent.

Lemma. *If G is a finitely generated, torsion free and nilpotent group then G is isomorphic to the $n \times n$ Heisenberg group.*

Free Products

The free product is the coproduct in the category of groups. i.e. The free product of A and B is a group $L = A * B$ with two homomorphisms $\iota_A : A \to L$ and $\iota_B : B \to L$ such that for any group G and morphisms f,g there is a unique arrow making the diagram commute



Given two groups $A = \langle X|R\rangle$ and $B = \langle Y|S\rangle$ then $A * B \cong \langle X \cup Y|R \cup S\rangle$. i.e. the free prduct is the smallest group that contains both A and B but does not require any new relations.

We define a normal form on the elemts of the free product. We say that an element of $A * B$ is a reduced alternating word iff it ahs the form

$$w = a_1 b_1 a_2 ... a_n b_n$$

such that $a_1 \in A, b_n \in B, a_i \in A \setminus \{1\}$ and $b_i \in B \setminus \{1\}$. The empty word is the identity.

Lemma. *Every element of the free product of two groups has a unique alternating word*

Proof. Existence is clear from the presentation (the letters generate). Now show uniqueness. Denote the set of all reduced alternating words Ω and the symetric group (group of automorphism of Ω) b yS_Ω.

Consider the homomorphisms $i : A \to S_\Omega, j : B \to S_\Omega$ given by concatenation and then reduction. Then from the free product definition we get a homomorphism extension

$$\phi : A * B \to S_\Omega$$

Such that for any $w \in \Omega$ we have that $\phi(\bar{w})(\epsilon) = w$ (for the empty word ϵ) where $\bar{w}$ is the group element given by the word w. Then if $\bar{u} =_{A*B} \bar{v}$ are group elements represented by words u, v

$$u = \phi(\bar{u})(\epsilon) = \phi(\bar{v})(\epsilon) = v$$

Some corraralleries are

- If $g \in A * B$ has normal form $g = a_1...b_n$ where $a_1 \neq 1$ and $b_n \neq 1$ then g has infinite order
- If both A and B are nontrivial then their free product contains an element of infinite order
- If an element $g \in A * B$ has finite order then it is conjugate to an element of A or B

Proof is exercise (1 and 2)

Theorem. *If $H \leq A_1 * ... * A_n$ then there is some free group F and H_i's such that*

$$H = F * H_1 * \cdots * H_n$$

such that each H_i is conjugate to a subgroup of some A_j

Note that all of the H_i in the above could be conjugate to subgroups of the same A_j or all different, there is no restriction.

Lemma. *A group cannot be expressed as a (nontrivial) free product and direct product*

Proof. (Sketch.)
Assume that $G \cong A * B \cong C \times D$. Now if $A \cap C \neq \{1\}$ then

$$\begin{aligned} & C_G(A \cap C) \leq A && \text{free product property (assignment 3)} \\ \Longrightarrow\ & D \leq A && \text{(everything in D commutes with everything in C)} \\ \Longrightarrow\ & C \leq C_G(D) \leq A \\ \Longrightarrow\ & G = A \end{aligned}$$

This contradicts our non-triviality assumption (in particular of B). Using the same method for the other combinations we can conclude that $A \cap (C \cup D) = 1$ and $B \cap (C \cup D) = 1$. Hence $C \leq A * B$. Using a lemma we know that

$$C = F * C_1 * \cdots * C_N$$

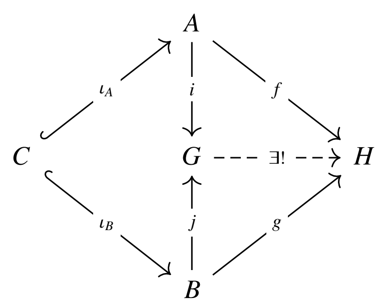
however we know that $N = 0$ otherwise $C \cap (A \cup B) \neq 1$. This implies that both C and D are free. We also know that

$$A \cong A/(A \cap C) \cong AC/C \leq G/C \cong D$$

Hence A is a subgroup of a free group and therefore free, likewise for B. Therefore G is free. A contradiction.

Amalgamation

We combine two groups and identify them along an isomorphic subgroup. A group G is a free product with amalgamation of A and B over C, with injections $\iota_A : C \to A$ and $\iota_B : C \to B$, denoted $A \underset{C}{*} B$ if there are arrows $i : A \to G$ and $j : B \to G$ such that G is universal in the diagram



In terms of presentations, if $A = \langle X_A | R_A \rangle$, $B = \langle X_B | R_B \rangle$ and $C = \langle X_C | R_C \rangle$ then we have that

$$A \underset{C}{*} B = \langle X_A, X_B | R_A, R_B, \{\iota_A(x) = \iota_B(x) : x \in X_c\} \rangle$$

We can fix a normal form for the amalagamated free product. First fix a set $T_A \subset A$ of coset representatives for $\iota_A(C)$ in A, likewise a set $T_B \subset B$, then each elemtn can be uniquely written in the form

$$a_1 b_1 a_2 \cdots a_n b_n c$$

where $a_i \in T_A, b_i \in T_B, c \in C$.

HNN Extensions

If we have a group presentation $G = \langle X | R \rangle$ and isomorphic subgroups $\phi : A \xrightarrow{\sim} B$ then the HNN extension is the group

$$G \underset{\phi}{*} \circlearrowleft = \langle X, t | R, \{t^{-1} a t = \phi(a) : a \in A\} \rangle$$

Note that adding $\{t^{-1} a t = \phi(a) : a \in X_A\}$ for some generating set X_A for A is enough.

Again HNN extensions have a normal form, if we first choose coset representatives $Y \subset G$ for A in G and $Z \subset G$ for B in G then every element of $G \underset{\phi}{*} \circlearrowleft$ can be represented by

$$g = g_1 t^{\epsilon_1} \cdots g_n t^{\epsilon_n} g_{n+1}$$

where $g_i \in G, \epsilon_i = \pm 1$ moreover if $\epsilon_i = 1$ then $g_i \in Y$ or if $\epsilon_i = -1$ then $g_i \in Z$, finally there are no pinches, subwords of the form $t^{-1}xt$ for $x \in A \cup B$.

Proof.

Proof

Lemma. *Let $G, X, G \underset{\phi}{*} \circlearrowleft$ be a HNN extension (X is the generating set of G). Then*

- *The identity map on X induces an injection $G \to G \underset{\phi}{*} \circlearrowleft$*
- *If $w \in (X \cup \{t\})^*$ some word, if $w =_{G \underset{\phi}{*} \circlearrowleft} 1$ then it contains a pinch.*

Theorem. *Any countable group can be embedded in a 2 generator group.*

Proof. Let the countable group be $C = \langle c_1, ... | D \rangle$, with some countable presentation. Let $F = \langle a, b | - \rangle$. We will embed $C \hookrightarrow F$.

Let $B = \langle a, b^{-1}ab, ..., b^{-i}ab^i \rangle \leq F$ (free as a subgroup of a free group) and notice that this list of generators is a free basis for B. Let $L = C * F$ and finally $A = \langle b, c_1 a^{-1} ba, ..., c_i a^{-i} ba^i, ... \rangle \leq L$. A is free on the generators listed there.

Prove that the two sets listed as free bases are free bases

Let

$$\phi : A \to B, \begin{cases} a \mapsto b \\ b^{-i}ab^i \mapsto c_i a^{-i} ba^i \end{cases}$$

Be the homomorphism identifying these generating sets. Now the HNN extension

$$G = L \underset{\phi}{*} \circlearrowleft = \langle a, b, t, c_1, ... | D, t^{-1}at = b, t^{-1}b^{-i}ab^i t = c_i a^{-i} ba^i \rangle$$

is the required group. It is clear that we can eliminate the c_i using Teitsze transformations giving some presentation

$$G \cong \langle a, b, t | D, t^{-1}at = b \rangle$$

But then again we can Tietze transformations to remove the relation and generator containing b. Moreover it is clear that $C \hookrightarrow L \hookrightarrow L \underset{\phi}{*} \circlearrowleft$ So we are done.

Theorem. *There are uncountably many (non-isomorphic) 2 generator groups*

Proof. Let $\sigma \subset \mathbb{N}$ a collection of primes. Define

$$A_\sigma = \prod_{p \in \sigma} \mathbb{Z}/p\mathbb{Z}$$

(the direct product). Then A_σ contains an element of order p iff $p \in \sigma$. Therefore $A_\sigma \cong A_\tau$ iff $\sigma = \tau$. Each A_σ embedds in a two generator group of the form $G_\sigma = A_\sigma * F_2 * \circlearrowleft$, hence any finite order element of G_σ must be conjugate to an element of A_σ. So $G_\sigma \cong G_\tau$ iff $\sigma = \tau$. There are uncountably many sets of primes so we are done.

Theorem. *Let A and B be finitely presented groups then*

- *$A \underset{C}{*} B$ is finitely presented iff C is finitely generated*
- *$G \to G \underset{C}{*} \circlearrowleft$ is finitely presented iff C is finitely generated*

Decision Problems

A collection $\mathcal{L} \subseteq X^*$ of words over a finite alphabet X is recursively enumerable if there is a Turing machine that enumerates the elements of $\mathcal{L}$. A set of words is recursive if both $\mathcal{L}$ and $X^* \setminus \mathcal{L}$ are recursively enumerable (r.e.).

There are recursively enumberable sets that are not recursive. The classic example is the set of halting Turing machines. Given a finite presentation $G = \langle X|R\rangle$ then the set $\mathcal{L} = \{x \in F(X) : w =_G 1\}$ is (r.e.), it is recursive iff $\mathcal{L}$ is recursive.

A group is recursively presentable iff it has a presentation $\langle X|R\rangle$ such that X is finite and R is a recursively enumerable set.

Theorem. *A finitely presented group always has a recursively enumberable set of trivial words*

Proof. A word $w =_G 1$ iff w is a product of conjugates of the relators

$$w = \prod^N g_i r_i g_i^{-1}$$

for $g_i \in F(X)$ and $r_i \in R$ the set of relators.

We can enumerate all the terms when $N = 1$ or 2 etc. Then we construct an algorithm by enumerating "one at a time" in a zigzag way, similar to how the rationals are shown to be countable.

Theorem. *There are recursively enumerable subsets of* $\mathbb{N}$*that are not recursive*

Proof. (Sketch.) We encode $\mathbb{N}$into the language of the Turing machine by looking at its alphabet $X = \{s_1, ...\}$ and defining $n = s_1...s_1 = s_1^n$ i.e. n occurences of s_1 next to each other. It is possible to ennumerate all Turing machines. So let $T_1, T_2, ...$ be such an enumeration.

$$E = \{n \in \mathbb{N} : T_n \text{ accepts } s_1^n \in X^*\}$$

Claim that E is recurrsively enumerable but not recursive.

Recursively enumerable is the same kind of argument as showing the rationals are countable.

Now imagine that $E' = \mathbb{N} \setminus E = \{n \in \mathbb{N} : T_n \text{ does not accept } s_1^n\}$ is also recursively enumerable. Then there is some m such that T_m accepts E'. But

$$m \in E' \iff m \in \mathcal{L}(T_m) \iff T_m \text{ accepts } s_1^m \iff m \in E$$

Hence E' is not r.e. and hence E is not recursive.

Theorem. *There are finitely presented groups for which it is undecidable if a given word is the identity*

Proof. Fix an $E \subset \mathbb{N}$ that is r.e. but not recursive. Let

$$G = \langle a, b, c, d|\{a^{-i}ba^i = c^{-i}da^i : i \in E\}\rangle = F_2 \underset{K}{*} F_2$$

where $K_1 = \langle\{a^{-i}ba^i : i \in E\}\rangle$ and $K_2 = \langle\{c^{-i}dc^i : i \in E\}\rangle$. Now if G has a solvable word problem then E is recursive (a contradiction). Why:

$\forall m \in \mathbb{N},\ \ a^{-m}ba^m =_G c^{-m}dc^m$ iff $m \in E$. $\Leftarrow$ is immediate because it is an explicit relator. $\implies$ can be seen by considering the normal form in the amalgamated free product of

$$(a^{-m}ba^m)(c^{-m}d^{-1}c^m) =_G 1$$

The uniqueness of the normal form tells us that if we put the above into normal form then it must be the empty word but for this to be the case it must be that $a^{-m}ba^m \in K_1$ and this is only possible when $m \in E$, likewise for the other.

Theorem (Higmans Theorem)**.** *A finitely generated group G is isomorphic to a subgroup of a finitely presentable group iff G is recursively presentable.*

Theorem. *There exists a finitely presentable group that contains an isomorphic copy of every finitely presented group*

Markov Properties

A property, P, of a finitely presentable group is called Markov if there is some group having that property and there is a finitely presented group G such that if G embedds in a group H then H does not have the property.

Theorem (Adian-Rabin)**.** *Markov properties are undecidable.*

Proof. Let M be a Markov property, and let P be the finitly presented "poison" group (as in the definition). Let U be a finitely presented group with undecidable word problem and finally let N be the f.p. group with property M.

Consider the free product $U * P$, a group that has undecidable word problem and does not have property M (contains both U and P). Fix a finite presentation

$$U * P = \langle x_1, ..., x_m | R\rangle$$

The strategy is to show given a word $w \in (X \cup X^{-1})^*$ where $X = \{x_1, ..., x_m\}$ then there is a group G_w such that G_w has M iff $w = 1$.

Now to construct G_w as follows: Let $U_1 = U * P * \langle y_o | - \rangle$, let $y_i = y_0 x_i$. Then notice that $\{y_0, ..., y_m\}$ generates U_1 and that the order of y_i is infinite.

Next let $U_2 = \langle U_1, t_0, ..., t_m | t_i^{-1} y_i t_i = y_i^2\rangle$ (a sequence of HNN extensions). Then $U_3 = \langle U_2, z | z^{-1} t_i z = t_i^2\rangle$ (stable letters in HNN extensions have infinite order). $V_3 = \langle r, s, t | s^{-1} r s = r^2, t^{-1} s t = s^2\rangle$. Notice that if t in V_3 is made to be the identity (in a quotient say) then each of the generators in turn will be forced to be the identity. Finally let

$$W = \langle U_3, V_3 | z = r, [w, y_0] = t\rangle$$

Note that $\langle r, t\rangle \cong F_2$ as the subgroup of V_3.

If $\langle z, [w, y_0]\rangle \cong F_2$ we would have that W is the amalgamated free product. If $w \neq 1 \in U * P$ then $\langle z, [w, y_0]\rangle \cong F_2$ hence W is an amalgamated free product and so $P \subseteq U_1 \subseteq U_2 \subseteq U_3 \subseteq W$ hence W does not have property M.

If $w = 1$ then $[w, y_0] = 1$ and we see a cascade of trivilisations, we get $t = 1$ hence $s = r = z = 1$ so all of the $t_i = 1$, so the image of U_1 in W is the trivial group, same with U_2 and so $W = 1$.

Let

$$G_w = W * M$$

And notice that this group has property M iff W is the identity and we are done.

Given a finite presentaiton of a group it is not possible to decide if the group is

- Trivial
- Finite
- Abelian
- Solvable
- Free
- Torsion free
- Simple (no proper normal subgroup)
- Decidable word problem

Hyperbolic Groups

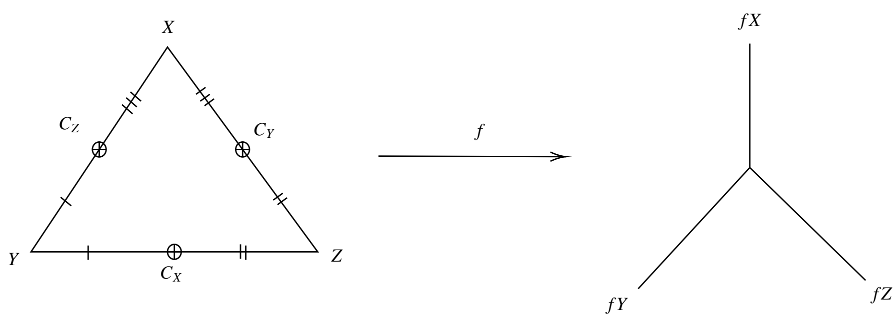
Hyperbolic Metric Spaces

Recall that a metric space is geodesic iff for every two points there is a path that has the length of the distance between the points. (A path being an isometric embedding of a connected subset of $\mathbb{R}$to the space).

Definition:

- *A geodesic triangle in a metric space is simply three geodesic segments denoted* $[x, y], [y, z], [z, x]$
- *A geodesic triangle is* $\delta-$*slim if each side is contained within a* δ *neighbourhood of the other two i.e. for each side*
  $$\forall p \in [x, y]\ \exists q \in [x, z] \cup [z, y] \text{ such that } d(p, q) \leq \delta$$
- *A geodesic triangle is* $\delta-$*thin if starting at a vertex and moving simultaneously along two of the seperate paths the distance between two of the points is at most* δ
- *A geodesic metric space is thin if every geodesic triangle is thin (some* δ*)*
- *Slim if every geodesic triangle is slim*
- *A function* $e : \mathbb{N} \to \mathbb{R}$ *is a divergence function of some metric space X iff*
  - *for all* $x \in X$*, all geodesics* $\gamma = [x, y], \eta = [x, z]$*, all* $r, R \in \mathbb{R}$ *and all paths* $p : I \to X \setminus B_{R+r}(x)$
  - *if* $r + R \leq min\{d(x, y), d(x, z)\}$*,* $d(\gamma(R), \eta(R)) > e(0)$ *and p is between* $\gamma(R + r)$ *and* $\eta(R + r)$
  - *Then the length of p is strictly greater than* $e(r)$
- *A hyperbolic metric space is one with exponential divergence of geodesic rays (geodesics* $r, r' : [0, \infty) \to X$ *that start at the same point)*
- *A quasi-geodesic is a quasi-isometric emedding of an interval into a metric space.*
- *Two geodesics u, v k-fellow travel iff* $\forall t \in \mathbb{N}$ *we have that* $d(u(t), v(t)) \leq k$

Lets make the definition of slim precise. It is a linear algebra fact that given a geodesic triangle there are points each edge equidistant from the verticies. There is a unique continuous map f from the triangle to a tripod with edge lengths exactly $d(X, C_Y), d(Y, C_Z), d(Z, C_X)$ sending C_X, C_Y, C_Z ot the center. We call the triangle $[x, y, z]$ $\delta-$thin if for all points $p \in f([x, y, z])$ the preimage $f^{-1}(p) \subseteq X$ has diameter at most δ.



Theorem. *A geodesic metric space is thin iff it is slim*

Proof.

Theorem. *A geodesic metric space is hyperbolic iff it is thin*

A quasi-geodesic need not be continuous however in a geodesic metric space we can always find a nearby continuous path. If γ is the image of a quasi-isometric embedding $\phi : I \to E$ then we can divide I into degments of length at most 1 and let $x_0, ..., x_n$ be the points of γ corresponding to the ends of these segments.

Build a path P by joining these points by geodesics in E, with each geodesic having length

$$d(x_i, x_{i+1}) \leq \lambda + \epsilon$$

If $t \in \gamma$ we get that $d(t, x_i) \leq \lambda + \epsilon$ for some i and if $t \in P$ then it is withing $\lambda + \epsilon$ of γ. Hence

$$d(\gamma, P) \leq \lambda + \epsilon$$

We use this construction in the next theorem

Theorem. *In a hyperbolic metric space there is some $R \geq 0$ depending only on $\delta, \lambda, \epsilon$ such that for any λ, ϵ-quasigeodesic ξ and any geodesic γ with the same endpoints*

$$d(\xi, \gamma) \leq R$$

Proof. Let the exponential divergence function for the metric space be e and define $D = sup_{x \in \gamma}\{d(x, \xi)\}$. We just need to show that D is bounded.

We use our construction to assume that ξ is continuous hence the supremum is realised at some $p \in \gamma$. Hence the interior of $B_D(p)$ does not intersect ξ (because $d(x, \xi) = \inf_{u \in \xi} d(x, u)$).

Let $a, b \in \gamma$ with distance D from p and $a', b' \in \gamma$ distance $2D$ from p. (if γ is too short it doesnt matter because we bound above.) D is the sup so there are points $u, v \in \xi$ that are within a', b' respectively. So the path $[u, a', a, b, b', v]$ has length at most $6D$. ξ being a quasi-geodesic tells us that the subpath from u to v has length at most $6\lambda D + \epsilon$ hence a path outside $B_D(p)$ of length at most

$$6\lambda D + \epsilon + 4D$$

Then from the definition of the divergence function

$$e(D - \frac{e(0)}{2}) < 6\lambda D + \epsilon + 4D$$

But e is exponential and so cant be bounded for all D (making it bounded by a linear function) hence this must be true only for a fixed (finite) D.

i.e. In hyperbolic space quasi-geodesics stay close to geodesics

Theorem. *If $f : X \to Y$ is a quasi-isometric embedding of geodesic metric spaces and Y is hyperbolic then X is hyperbolic.*

Proof. Let Δ be a geodesic triangle in X. Then applying f we get a quasigeodesic triangle $f(\Delta)$ in Y. Replace the edges of $f(\Delta)$ with geodesics between the verticies to get the geodesic triangle in Y Δ_2. Y is hyperbolic so Δ_2 is δ-slim.

If p is a point of the side of Δ then $f(p)$ is within R of a point of Δ_2 which using slim is within δ of the other geodesic sides so$f(p)$ is within $2R + \delta$ of the other quasi-geodesic side. (R is some constant from the previous lemma). Hence there is a point $f(q)$ on the quasi-geodesic

$$d_Y(f(p), f(q)) \leq 2R + \delta$$

Hence

$$d_X(p, q) \leq \lambda d_Y(f(p), f(q)) + \epsilon \leq \lambda(2R + \delta) + \epsilon$$

Which is a constant (depending only on the constants of $\lambda, \epsilon, \delta$).

Hyperbolic Groups

Definition:

- *A finitely generated group is called hyperbolic iff its Cayley graph is hyperbolic for some (every) finite generating set*
- *A finite presentation $\langle X|R\rangle$ for the group G is calleda Dehn presentation iff for any $w \in F(X)$ such that $|w| \geq 1$ and $w =_G 1$ there is some $r \in R$ such that $r = uv$ where u is a factor of w and $|u| > |v|$*
- *A path is k-local geodesic iff every connected subpath of length at most k is a geodesic*

Lemma. *Hyperbolic groups are finitely presentable*

Proof. (Sketch.)
By assumption the group is finitely generated and hyperbolic. So fix a finite generateing set X and a $\delta \geq 1$ such that the triangles in $Cay(G, X)$ are $\delta-$thin. Then let

$$R = \{r \in F(X) : r =_G 1, |r| \leq 2\delta + 2\}$$

Then one can show that all the loops in the Cayley graph can be filled in using cells of perimeter at most $2\delta + 2$ hence $\langle X|R\rangle$ is a finite presentation.

Theorem. *G a hyperbolic group with generation set S such that $Cay(G, S)$ has $\delta-$thin triangles. If $k \geq 8\delta$ then every k-local geodesic is a λ, ϵ-quasigeodesic where the λ, ϵ are functions of δ and k.*

Theorem. *A group is Hyperbolic iff it admits a Dehn presentation*

Proof. Let G be generated by a finite set X and let δ be such that the triangles in $Cay(G, X)$ are $\delta-$thin. Let

$$R = \{r \in F_X : r =_G 1, |r| \leq 8\delta\}$$

Claim that $\langle X|R\rangle$ is a Dehn presentation for G.
Now take a $w \in F_X$ such that $w =_G 1$. We want a relator $r = uv$ where $|u| > |v|$ and u is a subword of w. If $|w| \leq 8\delta$ we let $r = u = w$ (we explicitly added it as a relator). So we may assume that $|w| > 8\delta$.
If w is not a 4δ local geodesic then there is a sub-word u of w which has length $\leq 4\delta$ which is not geodesic. Then let v geodesic such that $uv =_G 1$ and $|v| < |u|$ moreover $uv \in R$.
If w is a 4δ local geodesic then it is a quasigeodesic from 1 to 1 hence has bounded length.

An isoperimetric funciton for a presentation is a monotone non-decreasing function from $\mathbb{N}$to the positive reals such that for any word reducing to the identity its area is bounded above by the function.

Theorem. *If a finitely presentable group satisfies a linear isoperimetric inequality iff it is hyperbolic.*

Proof

exercise on page 7, lecture 3

Theorem. *A hyperbolic group has, up to conjugacy, a finite number of finite subgroups.*

Proof. Let $H \leq G$ a finite subgroup. Let $X = Cay(G, S)$, $C = c_1(H)$ and $Y = H$ in the result from the exercise.
There is at least one vertex $g \in C$ and $Diam(C) \leq 2(\delta + 1)$. THe set C is invariant under the action of H. Then we get $1_G \in g^{-1}C$ and

$$g^{-1}Hg \subseteq g^{-1}C \subseteq B_{2\delta+2}(1_G)$$

Hence some conjugate of H lies in the ball in the Cayley graph, but there are finitely many subgroups contained in a fixed ball.

Definition: *A subset $Y \subseteq X$ of a geodesic metric space is called $k-$quasiconvex iff every geodesic in X that has both engpoings in Y is contained in the $k-$neighbourhood of Y.*

Lemma. *If $H \leq G$ is a quasiconvex subset of $Cay(G, S)$ (S a finite generating set for G) then H is finitely generated and $H \hookrightarrow G$ is a quasi-isometric embedding*

Proof. Let H be quasi convex and define $T = \{h \in H : d_S(1, h) \leq 2k + 1\}$, a finite set. Claim that T generates H.
If $h \in H$ then $h = s_1 \cdots s_n$ for some shortest word over S. Then each vertex along w

h= Gw?

Lemma. *For a finitely generated subgroup $H \leq G$ of some hyperbolic group then H is quasiconvex (in the Cayley graph) iff $H \hookrightarrow Cay(G, S)$ is a quasiisometric embedding. Moreover if H is quasiconvex in some Cayley graph then it is quasiconvec in any.*

Lemma. *Quasiconvex subgroups of hyperbolic groups are hyperbolic*

Proof. For a fixed finite generating set S of G and T for H there is a quasi-isometric embedding $Cay(H,T) \hookrightarrow cay(G,S)$ hence by a corrollery H is hyperbolic (embed into hyperbolic implies hyperbolic).

Lemma. *If $K < H \leq G$ a finitely generated group and H and K are quasiconvex then $H \cap K$ is quasiconvex.*

Lemma. *The centraliser of an element in a hyperbolic group is a quasiconvex subgroup.*

Proof. Fix a finite generating set X and an element $g \in G$. If you take two elements $a, b \in C(g)$ and a geodesic between them then by applying the isometry on $Cay(G,X)$ induced by multiplicaiton by a^{-1} gives a geodesic $\gamma = x_1 \cdots x_n$ between 1 and $a^{-1}b$. Hence if we can prove that every point on $\gamma$is within a bounded distance of $C(g)$ we will be done.

$\gamma \in C(g)$ tells us that $g\gamma g^{-1} = \gamma$and that there is some $K = max(\delta, d(1,g))$ such that $d(\gamma_i, g\gamma_i) \leq K$ (one can see this by considering the fellow traveling path of g then γ or the other way (as a triangle), and the property of thin).

Now let $M = |B_K(q)|$. Claim that for all i there is an element $\eta_i \in C(g)$ that is within M of γ_i.

If $n - i \leq M$ then $\eta_i = \gamma$.

If $n - i > M$ then there are $k > j > i$ such that $\gamma_j^{-1} g \gamma_j = \gamma_k^{-1} g \gamma_k$

why...

Then $x_1 \cdots x_j x_{k+1} \cdots x_n \in C(g)$, this is strictly shorter than the original word with the same prefix, so we can repeat until $n' - i \leq M$ and so we are done.

Nah lost me at the end

Proof questions

Lemma. *The center of a hyperbolic group is quasiconvex and virtually cyclic.*

Proof. G is finitely generated by X then the center of G being quasi convex follows from Howsons lemma and the fact that

$$Z(G) = \Cap_{x \in X} C(x)$$

since the center is quasiconvex in G it is hyperbbolic. Since it is both hyperbolic and abelian (by an exercise) it is virtually cyclic.

Lemma. *If G is a hyperbolic group and $g \in G$ has infinite order then*

- *$\mathbb{Z} \to G$ sending $m \to g^m$ is a quasi-isometric embedding*
- *The grouo generated by g is finite index in $C(g)$*

Proof. X finite generating set δ-thin cayley graph.

uses some constnats and stuff from previous proofs quite tedious I think

Theorem. *A hyperbolic group cannot contain a subgroup isomorphic to $\mathbb{Z}^2$.*

Proof. Suppose that $H \leq G$ and that $H \cong \mathbb{Z} \times \mathbb{Z}$ and G is hyperbolic. Then any element $h \in H \setminus \{1\}$ gives that $\langle h \rangle$ is finite index in $C(h)$.

Then $\langle h \rangle \leq H \leq C(h)$ because H is abelian so $\langle h \rangle$ must be finite index in H, so H is virtually cyclic, this is a contradiction.

I dont understand the last two steps.

We note that this can be used as the "poison" group in the demonstration of a property being Markov, hence being hyperbolic is a Markov property.

Visual Boundary

Two geodesic rays $r, r' : [0, \infty) \to X$ in some metric space X are asymptotic iff $sup_t d(r(t), r'(t)) < \infty$.

The boundry of X, denoted ∂X is the set of all geodesic rays modulo this equivilence relation. The equivilence class of ray r is denoted $r(\infty)$.

Lemma. *If X is a proper (closed balls are compact), hyperbolic and geodesic metric space then for each point on the boundry $\xi \in \partial X$ and each point $p \in X$ there is some geodesic ray r such that $r(0) = p$ and $r(\infty) = \xi$.*

i.e. we can base our rays at any point in a hyperbolic, geodesic and proper metric space.

When S is a finite generating set for a hyperbolic group G and $X = Cay(G, S)$ then there are precisely three possibilities for the cardinality of the boundry of X:

$$|\partial X| = \begin{cases} 0, & \text{iff G is finite} \\ 2, & \text{iff G is virtuallly } \mathbb{Z} \\ |\mathbb{R}|, & \text{otherwise} \end{cases}$$

Decidability in Hyperbolic Groups

Lemma. *Hyperbolic groups have linear solutions to the word problem*

proof

The conjugacy problem for a fintely generated group is deciding if a given element can be reached from another given element by conjugajing by something in the group.

Exercises

Unfinished

- Prove that $Cay(G, S)$ is diconnected iff S is a monoid generating set for G
- Prove that if there is a path in Γ from g to h adn the label of the path is $t_1...t_n$ then $h = gt_1...t_n$ and $t_1...t_n$ spells $g^{-1}h$ (the prodcut evaluates in the group to $g^{-1}h$)
- Show that $SL(2, \mathbb{C})$ contains a free subgroup (hints given)
- The subgroup of F_2 with free basis $\{a, b\}$ generated by $S = \{a^i ba^{-i} : 0 \leq i \leq n-1\}$ is free of rank n.
- Ping pong lemma exercise 23 apply to some annoying to write group (copy out question)
- Prove that if $a^{-1}b^{-1}ab \in R$ then $ba^{-1}b^{-1}a, b^2ab^{-2}a^{-1} \in \langle R\rangle$
- Prove that if w is a word in the letters a, a^{-1}, b, b^{-1} with equal numbers of b and b^{-1} then $w \in \langle aba^{-1}b^{-1}\rangle$
- Braid group exercise 26 (write up)
- Cayley Graphs of group presentations (E 27) (write up)
- Prove $\langle a, b|a^{-1}ba = b^2, b^{-1}ab = a^2\rangle$ is the trivial group
- $(\mathbb{Z}, +)$ can be generated by $\{1\}$ or $\{2, 3\}$. Use multiplicative notation to write down presentations for both generating sets then use Tietze transformations to connect them.
- (Exercises 31,32,33)
- Prove the following facts about the word length
  - $||g||_s = 0 \iff g = 1$
  - $||g||_S = ||g^{-1}||_S$
  - $||g_1g_2||_S \leq ||g_1||_S ||g_2||_S$
- Check that the d_S is a metric
- Being isometric is an equivilence relation on metric spaces
- The set of isometries on a metric space forms a group under composition
- Show explicitly that $\langle a, x|ax = xa, x^2\rangle \sim \langle b|-\rangle$ (q.i.)
- Show explicitly that $\mathbb{R}^2 \sim Cay(\mathbb{Z}^2, \{a, b\})$ (q.i.)
- exercise 16 and 17
- Being $\lambda - biLipschitz$ equivilent is an equivilence relation on metric spaces
- Being quasi-isometric is an equivilence relation on metric spaces
- Compute the growth function for $\mathbb{Z}$generated by $\{2, 3\}$; How much does changing the generating set change the growth function?
- $$f_{D_\infty,\{s,t\}}(n) = 4n$$

- Check that $\preceq$ is an equivilence relation on monotone increasing functions $\mathbb{N} \to \mathbb{N}$
- Use submultiplicativity to show that $\omega(G, S)$ is well definied. Show that the limit exists and in fact is equal to $inf(f_{G,S}(n))^{\frac{1}{n}}$
- exercises 38 - 43 (section 6) and section 7 (Dehn functions)
- Show that the arrows ι_A and ι_B are injective.
- Show the free product is unique up to isomorphim
- Draw the Cayley graph of the free products $C_2 * C_2$ and $C_2 * C_3$
- Show the amalgamated free product is unique up to isomorphism
- Show that arrows i, j in the amalgamated free product are injective.
- trefoil knot exercise 7 and 8
- Write $t^{-10}at^{10}$ in normal form as an element of $BS(1, 2) = \mathbb{Z} \underset{\phi}{*} \circlearrowleft$ where $\phi : \mathbb{Z} \to 2\mathbb{Z}, 1 \mapsto 2$ (Check this HNN extension of BS)
- Prove that $C_2 * C_3$ contains a free subgroup of finite index (the property of being virtually free).
- Show that in an amalgamated free product any element of finite order is conjugate to an element of one of the factors.
- A word is cyclically reduced iff applying a cyclic permutation to it does not make it unreduced. Show that two cyclically reduced words in a free group are conjugate iff one is a cyclic permutation of the other.
- Prove that the Heisneberg group and $BS(1, 2)$ (exercise 26) are linear.
- Show that if G, H are hyperbolic then so is the free product
- wednesday week 4 slide 7
- If X is $\delta-$thin and r, r' are asymptotic starting at the same point then $sup_t d(r(t)mr'(t)) \leq \delta$

Ok go through move the completed ones (done typed or written) to the next section, type stuff up and just keep going though them etc add the rest of teh questions from the course

Cayley Graphs and Presentations

1. Prove an undirected graph is a tree iff for every pair of verticies there is a unique path between them without loops or backtracking

Proof. ($\Longrightarrow$)Let Γ be a tree, i.e. connected with no non-trivial cycles. Let $u, v \in V$. From connectedness we get a path $u = a_0, ..., , a_n = v$ in Γ. We can assume without loss of generality that there are no loops or backtracking (simply remove those parts if there is and we still have a path).

To show uniqueness it will suffice to consider the case where there are two paths between vertices that agree nowhere (except start and finish). This is because in the general case we simply need to identify this subcase and apply the following argument.

So let $u = a_0, ..., a_n = v$ and $u = b_0, ..., b_n = v$ two paths such that $a_i \neq b_i$ for every $i \in \{1, ..., n-1\}$. But then $b_0, ..., b_n, a_{n-1}, ..., a_0$ is a nontrivial cycle and so we have a contradiction with Γ being a tree. Thus distinct paths cannot exist.

Virtually the same argument works for the reverse implication.

2. Show that $Aut(\Gamma)$, the set of all automorphisms of a graph is a group under function composition

Proof. Trivial

3. Draw Cayley graphs for the following

- $(\mathbb{Z}/20\mathbb{Z}, +)$ wrt $S = \{1\}$
- $(\mathbb{Z}/20\mathbb{Z}, +)$ wrt $S = \{1, 10\}$
- $(\mathbb{Z}, +)$ wrt $S = \{2, 3\}$
- $(\mathbb{Z}^2, +)$ wrt $S = \{(1, 0), (0, 1)\}$
- $(\mathbb{Z}^2, +)$ wrt $S = \{(1, 1), (0, 1)\}$
- $(\mathbb{Z}, +) \oplus (\mathbb{Z}/2\mathbb{Z}, +)$ wrt $S = \{(1, 0), (0, 1)\}$
- D_∞ the group of symetries of the integer line, wrt $S = \{r, s\}$ where $r(n) = 1 - n$, $s(n) = -n$

4. Prove that $Cay(G, S)$ is connected iff S is a generating set for G

Proof. Cay(G,S) is connected implies that there is a path between the vertex for the neutral element and any other. i.e. For any $g \in G$ there is a sequence $s_0, ..., s_n$ such that

$$g = es_0 \cdots s_n = s_0 \cdots s_n$$

i.e. S is a generating set.

For the reverse direction we see that if S is a generating set then $g = s_0 \cdots s_n$ for every g. Thus We have a path from every vertex to the neutral element and therefore the graph is connected.

5. Prove $Cay(G, S)$ has loops iff $1 \in S$

Proof. Cay(G,S) has a loop implies that the following is an edge for some $g \in G$, (g, g). This imiplies that for some $s \in S$ we have $gs = g$ which implies that $es = e$ i.e. $e \in S$.

If $e \in S$ then for any $g \in G$ $(g, ge = g)$ is an edge.

6. Prove $Cay(G, S)$ has no directed cycles of length less than 3 iff $S \cap S^{-1} = \emptyset$

Proof. Consider the contrapositive statement

$$Cay(G, S) \text{ has a directed cycle of length less than 3} \iff S \cap S^{-1} \neq \emptyset$$

So assume that there is a directed cycle of length less than 3. i.e. there is some $g, h \in G$ such that $(h, g), (g, h) \in E$ or there is a loop (some g such that $(g, g) \in E$).

In the case of the loop by the previous exercise we know that $1 \in S$ and since 1 is its own inverse we are done.

The other case holds iff $h = gs_1$ and $g = hs_2$ for some $s_1, s_2 \in S$. i.e. $g = gs_1 s_2$ so $s_1 s_2 = 1$ and thus they are mutally inverse. This is iff $S \cap S^{-1} \neq \emptyset$

7. Prove that if G acts on a connected directed graph, regularly on the vertices, then the graph is a Cayley graph of G wrt some generating set

Proof. Let G act on $\Gamma = (V, E)$ regularly on the vertices. Consider the following map for a given $v \in V$

$$\phi_v : G \to V$$

$$g \mapsto gx$$

This is a bijection: Surjectivity is immediate by transitivity (regular) of the action. Freeness implies injectivity as follows

$$\begin{aligned} gx = hx &\implies g^{-1}gx = g^{-1}hx \\ &\implies x = g^{-1}hx \\ &\implies g^{-1}h = 1 \text{ because by freeness the stabiliser of x is trivial} \quad \implies g = h \end{aligned}$$

Let $S = \cup_{v \in V} Im(\phi_v^{-1})$. Then claim that $\phi_x^{-1} : V \to G$ is an isomorphism of $Cay(G, G)$ and Γ. We have shown that it is a bijection on the verticies so we only need to check that it preserves adjacency.

I thought I had an argument but I dont see it working anymore

8. $(\mathbb{Z}, +) \cong C_\infty = \{x^i : i \in \mathbb{Z}\}$. Show that $\{x^{-1}\}$ is a free basis, that $\{x^2\}$ is not and that $\{x, x^{-1}\}$ is not.

Proof.

- $\boldsymbol{\{x^{-1}\}}$ **:** Let $\phi(x^{-1}) = g \in G$ then $\bar{\phi} : \mathbb{Z} \to G$ is definied by $x^i \mapsto g^{-i}$. This is clearly a homomorphism. It is also clear that $x^{-1} \mapsto g^{-(-1)} = g$ so $\phi = \bar{\phi}|_{\{x^{-1}\}}$. Finally if $\bar{\psi}$ is another extension then

$$\bar{\psi}(x^i) = (\bar{\psi}(x^{-1}))^{-i} = g^{-i} = \bar{\phi}(x^i)$$

- $\boldsymbol{\{x^2\}}$ **:** Consider the map $x^2 \mapsto 1 \in \mathbb{R}$. Then there are two different homomorphism extensions from $\mathbb{Z}$, namely $\phi : \mathbb{Z} \to \mathbb{R}, x \mapsto 1$ or $\psi : \mathbb{Z} \to \mathbb{R}, x \mapsto -1$.
- $\boldsymbol{\{\}}$ **:** There is no group homomorphism $\phi : (\mathbb{Z}, +) \to (\mathbb{Z}, +)$ satisfying $x \mapsto 1, x^{-1} \mapsto 1$ namely because $1^{-1} = -1$.

9. The trivial group is a free group

Proof. Its free on the empty set. Axioms trivially satisfied.

10. Two free groups are isomorphic iff they have the same rank (Proof given in lecture)
11. Every finitely generated group is a quotient of a finite rank free group

Proof. Let G be finitely generated on n generators. Then it has the presentation

$$\langle x_1, ..., x_n | \text{ all words on the generators that are trivial in G} \rangle$$

hence it is a quotient of the free group of rank n.

This is not very formal

12. Prove that $\{a, b, c\}$ is or is not a free basis for the group $G = \langle a, b, c | c = ba \rangle$

Proof. We can see with our knowledge of Tietze transformation that this is infact free on 2 generators so $\{a, b, c\}$ is not a free basis. The question wanted us to consdier a map into a finite group however and so I give as another proof the set map

$$\phi : \{a, b, c\} \to \mathbb{Z}/2\mathbb{Z}$$
$$a, b, c \mapsto 1$$

Then there is no homomorphism extending this because that would require $\phi(a, b) = 1 + 1 = 0$ to be equal to $\phi(c) = 1$ i.e. $1 = 0$ a contradiction.

13. Prove or disprove: A free group is abelian iff it is infinite cyclic.

Proof. We showed the trivial group is free, and it is trivially abelian.

14. Use Tietze transformations to transform the presentation $\langle a, b, c | c^{-1}ac = b, c^{-1}bc = a, c^2 = 1 \rangle$ into $\langle a, c | c^2 = 1 \rangle$

Coarse Geometry

- quasi-isometries have quasi-inverses
- Prove by constructing an explicit quasi-isometry that $Cay(\langle x | x^2 \rangle, \{x\})$ is quasi-isometric to a point
- Explain why every finite group is q.i. to a point.
- A homomorphism between finitely generated groups, $\phi : G \to H$ is a quasi-isometry iff $|ker(\phi)| < \infty$ and $[H : im(\phi)] < \infty$
- Prove that the integral Heisenberg group is a group
- In realtion to the Heisenberg group. Let

$$a = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, b = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, c = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

  - H is generated by a,b,c
  - $[a, c] = [b, c] = 1, [a, b] = c$
  - The subgroups of H generated by any one of the a,b or c are isomorphic to $\mathbb{Z}$.
  - c is central in H. i.e. $ch = hc$ for every other $h \in H$
  - $H/\langle c \rangle \cong \mathbb{Z}^2$
  - $[H, H] = \langle c \rangle$
  - Any $h \in H$ can be written as $a^i b^j c^k$ for some $i, j, k \in \mathbb{Z}$. Moreover, this form is unique.
  - Show this normal form for H is not geodesic i.e. there is some $h \in H$ such that

$$||h||_S < |i| + |j| + |k|$$

- Let $i, j, k \in \mathbb{Z}$ be given, then show that

$$||a^i b^j c^k||_S \leq n \implies |i| + |j| \leq n,\ |k| \leq n^2$$

$$||a^i b^j c^k||_S \leq |i| + |j| + 6\sqrt{|k|}$$

  (some hints given for the last one)
- $\gamma_H \sim n^4$
- Prove if G is f.g. and G is q.i. to a point then G is finite
- Check that the growth function of F_2 is $2^2 3^{n-1}$ for $n \geq 2$
- Compute $\gamma_{\mathbb{Z}^2}$ and $\gamma_{\mathbb{Z}^3}$
- Every nilpotent group is solvable
- Prove the following about the commutator
  * $[G, G] \triangleleft G$
  * $G/[G, G]$ is abelian
  * $N \triangleleft G$ such that G/N implies that $[G, G] \leq N$. i.e.$G/[G, G]$ is the maximal abelian subgroup of G.

Free Products

- Prove that having a decidable word problem is Markov
- Prove that there is a finitely presented group with unsolvable conjugacy problem

Hyperbolic Groups

- If E is hyperbolic then every pair of geodesics that start at the same point and end distance at most 1 apart k-fellow travel (where k depends only on the constant δ that definies the hyperbolicity)
- Prove that $Cay(\langle a, b | ab = ba \rangle)$ is not hyperbolic
- Let X be the infinite cylinder. Define a metric by length of shortest path. show that this space is hyperbolic.
- Prove if a group has a Dehn presentation then its word problem can be solved in linear time
- Show that if a hyperboilic group is abelian then it is virtually cyclic

Chapter 3

Appendix

Topological Graph

The lecturer for the second week of the AMSI course (course geometry, growth etc) mentioned to me the alternate and in my opinion more intuitive notion of a topological graph.

definition

Todo list